The general form of supersymmetric solutions of N=(1,0) U(1) and SU(2) gauged supergravities in six dimensions

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ABSTRACT: We obtain necessary and sufficient conditions for a supersymmetric field configuration in the N=(1,0) U(1) or SU(2) gauged supergravities in six dimensions, and impose the field equations on this general ansatz. It is found that any supersymmetric solution is associated to an $SU(2) \ltimes \mathbb{R}^4$ structure. The structure is characterized by a null Killing vector which induces a natural 2+4 split of the six dimensional spacetime. A suitable combination of the field equations implies that the scalar curvature of the four dimensional Riemannian part, referred to as the base, obeys a second order differential equation. Bosonic fluxes introduce torsion terms that deform the $SU(2) \ltimes \mathbb{R}^4$ structure away from a covariantly constant one. The most general structure can be classified in terms of its intrinsic torsion. For a large class of solutions the gauge field strengths admit a simple geometrical interpretation: in the U(1) theory the base is Kähler, and the gauge field strength is the Ricci form; in the SU(2) theory, the gauge field strengths are identified with the curvatures of the left hand spin bundle of the base. We employ our general ansatz to construct new supersymmetric solutions; we show that the U(1) theory admits a symmetric Cahen-Wallach$_4 \times S^2$ solution together with a compactifying pp-wave. The SU(2) theory admits a black string, whose near horizon limit is $AdS_3 \times S_3$. We also obtain the Yang-Mills analogue of the Salam-Sezgin solution of the U(1) theory, namely $R^{1,2} \times S^3$, where the S^3 is supported by a sphaleron. Finally we obtain the additional constraints implied by enhanced supersymmetry, and discuss Penrose limits in the theories.

KEYWORDS: gauged supergravities, G-structures.

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1. Introduction

Chiral N=(1,0) U(1) gauged supergravity [1], [2] has received considerable attention both in the past [3], [4] and more recently [5]-[10], in part due to phenomenological interest in the remarkable $\mathbb{R}^{1,3} \times S^2$ solution found by Salam and Sezgin in [3]. Given this recent interest, it is natural to attempt a classification of the supersymmetric solutions of the theory, employing the powerful techniques first used in [11] and developed in [12]-[20]. We also, for the first time, apply this technique to a non-abelian gauged supergravity, namely the N=(1,0) gauged SU(2) theory in six dimensions.

The strategy we use is to assume the existence of at least one Killing spinor. Then we may construct from that spinor a one form and a triplet of three forms, which satisfy various algebraic and differential conditions which follow from the Fierz identities and the Killing spinor equation. We exploit these, and the supersymmetry variations of the other fermions in the theory, to deduce the most general form of the bosonic fields compatible with supersymmetry. The existence of a Killing spinor implies that most of the equations of motion are satisfied identically. We impose the remaining field equations on the general supersymmetric ansatz.

As we will see, the vector dual to the one form constructed from the Killing spinor is both Killing and null. This induces a natural 2+4 split of the six dimensional spacetime. A combination of the field equations and Bianchi identities of the three form field strength present in the theory implies that the curvature of the four dimensional Riemannian part, or base, must obey a second order differential equation. Solving this equation is the biggest obstacle we encounter; disappointingly, in the U(1) theory we have been unable to find a base which does not arise in known solutions which induces a non-singular six dimensional metric. However, starting from "known" bases we have been able to construct new six dimensional solutions. Furthermore this general procedure for finding supersymmetric solutions yields considerable geometrical insight into the form of the solutions. For example, under a broad class of conditions (precisely specified below) one may deduce that the base must be positive scalar curvature Kähler, and that the gauge field strength is given by the Ricci form of the base.

Our results for the $SU(2)$ theory demonstrate that one may usefully apply this general approach to nonabelian gauged supergravities. We have been able to exploit the geometry of Killing spinors of the theory, and in particular the fact that one can construct a triplet of two forms which are anti self-dual on the base and (again for a broad class of solutions) $SU(2)$ covariantly constant thereon. The existence of these forms for this class of solutions allows the identification of the gauge field strengths with the curvatures of the left-hand spin bundle of the base. One might hope that something similar could be achieved in other nonabelian gauged supergravities.

The plan of the rest of this paper is as follows. In section 2 we give a brief introduction to the theories we study, and in section 3 we obtain the most general supersymmetric ansatz for each. We impose the field equations on this ansatz in section 4, and in section 5 we discuss intrinsic torsion. In section 6 we construct examples of supersymmetric solutions solutions of both theories. In section 7 discuss solutions of the theories with enhanced

supersymmetry, and in section 8 we discuss the Penrose limits of the theories. We conclude in section 9.

2. The supergravities

We will work in mostly minus signature and adopt the conventions of [2]. All spinors of the theory are symplectic Majorana, ie

$$\chi^A = \epsilon^{AB} \bar{\chi}_B^T, \quad \bar{\chi}_A = (\chi^A)^\dagger \gamma_0. \tag{2.1}$$

The $Sp(1)$ indices are raised and lowered as

$$\chi^A = \epsilon^{AB} \chi_B, \quad \chi_A = \chi^B \epsilon_{BA}, \quad \epsilon^{12} = e_{12} = 1. \tag{2.2}$$

The field content of the $SU(2)$ theory is follows: the gravity multiplet e_μ^m, $\psi_{\mu L}^A$, $B_{\mu\nu}^+$, a tensor multiplet $B_{\mu\nu}^-$, χ_R^A, ϕ and an SU(2) gauge multiplet A_μ^a, λ_L^{aA}. The subscripts denote the chiralities of the fermions, and $\pm$ means the potentials have self and anti self dual field strengths. To translate the conventions of [3] to those employed here one must change the signature of the metric, switch from a Weyl spinor to a pair of symplectic Majorana spinors, and also make the replacements $(\kappa, \sigma, g) \to (1, \sqrt{2}\phi, g/2)$.

Until section 6 we will treat the $U(1)$ and $SU(2)$ theories in tandem. Unless explicitly stated otherwise all expressions given for the $SU(2)$ theory are valid for the $U(1)$ case provided that two of the gauge potentials and the associated field strengths are set to zero (and of course there is only one gaugino in the $U(1)$ theory). We define $G_3 = dB_2 + F_2^a \wedge A_1^a - \frac{g}{6} f^{abc} A_1^a \wedge A_1^b \wedge A_1^c$ and $F_2^a = dA_1^a + \frac{g}{2} f^{abc} A_1^b \wedge A_1^c$. The bosonic Lagrangian of the theory is

$$e^{-1}\mathcal{L} = -\frac{1}{4}R + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{12}e^{2\sqrt{2}\phi}G_{\mu\nu\sigma}G^{\mu\nu\sigma} - \frac{1}{4}e^{\sqrt{2}\phi}F_{\mu\nu}^a F^{a\mu\nu} - \frac{ng^2}{8}e^{-\sqrt{2}\phi}, \tag{2.3}$$

where $n = 1$ for $U(1)$ and $n = 3$ for $SU(2)$. The fermion supersymmetry transformations are

$$\delta\psi_\mu^A = (\nabla_\mu - \frac{1}{4}e^{\sqrt{2}\phi}G_{\mu\nu\sigma}^+ \Gamma^{\nu\sigma})\epsilon^A + gA_\mu^a T^{aA}{}_B\epsilon^B, \tag{2.4}$$

$$\delta\chi^A = i(-\frac{1}{\sqrt{2}}\Gamma^\mu\partial_\mu\phi - \frac{e^{\sqrt{2}\phi}}{12}G_{\mu\nu\sigma}^-\Gamma^{\mu\nu\sigma})\epsilon^A, \tag{2.5}$$

$$\delta\lambda^{aA} = -\frac{e^{\phi/\sqrt{2}}}{2\sqrt{2}}F_{\mu\nu}^a\Gamma^{\mu\nu}\epsilon^A + \frac{e^{-\phi/\sqrt{2}}}{\sqrt{2}}gT^{aA}{}_B\epsilon^B, \tag{2.6}$$

where the $SU(2)$ generators $T^{aA}{}_B$ are given by $T^{aA}{}_B = -\frac{i}{2}\sigma^{aA}{}_B$ and the superscripts $\pm$ denote the self and anti self dual parts of G. The supersymmetry parameter ϵ is left-handed, ie $\Gamma_7\epsilon = -\epsilon$, where

$$\Gamma_7 \equiv \Gamma_0\Gamma_1...\Gamma_5. \tag{2.7}$$

The bosonic field equations and Bianchi identities are

$$R_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi + e^{2\sqrt{2}\phi}(G_{\mu\sigma\lambda}G_\nu{}^{\sigma\lambda} - \frac{1}{6}g_{\mu\nu}G_{\alpha\beta\gamma}G^{\alpha\beta\gamma})$$

$$+ 2e^{\sqrt{2}\phi}(-F^a_{\mu\alpha}F^a_\nu{}^\alpha + \frac{1}{8}g_{\mu\nu}F^a_{\alpha\beta}F^{a\alpha\beta}) - \frac{ng^2}{8}e^{-\sqrt{2}\phi}g_{\mu\nu}, \tag{2.8}$$

$$\nabla^2\phi = \frac{1}{3\sqrt{2}}e^{2\sqrt{2}\phi}G_{\mu\nu\sigma}G^{\mu\nu\sigma} - \frac{1}{2\sqrt{2}}e^{\sqrt{2}\phi}F^a_{\mu\nu}F^{a\mu\nu} + \frac{ng^2}{4\sqrt{2}}e^{-\sqrt{2}\phi}, \tag{2.9}$$

$$dG = F^a \wedge F^a, \tag{2.10}$$

$$d\star(e^{2\sqrt{2}\phi}G) = 0, \tag{2.11}$$

$$dF^a = -g\epsilon^{abc}A^b \wedge F^c, \tag{2.12}$$

$$d\star(e^{\sqrt{2}\phi}F^a) = -2e^{2\sqrt{2}\phi}\star G \wedge F^a - ge^{\sqrt{2}\phi}\epsilon^{abc}A^b \wedge \star F^c. \tag{2.13}$$

Note that we have corrected the Einstein equation given in [2].

3. Necessary and sufficient conditions for supersymmetry

Now we will implement the first part of our strategy and obtain the general supersymmetric ansatz for our theories. Given a Killing spinor ϵ we may construct the nonzero bilinears

$$V_\mu\epsilon^{AB} = \bar{\epsilon}^A\Gamma_\mu\epsilon^B, \tag{3.1}$$

$$\Omega^{AB}_{\mu\nu\sigma} = \bar{\epsilon}^A\Gamma_{\mu\nu\sigma}\epsilon^B. \tag{3.2}$$

Forms of even degree vanish because ϵ is chiral. The Ω^{AB} are self dual and we define the real self dual forms X^a ($a = 1, 2, 3$) by

$$\Omega^{11} = -\frac{1}{2}(X^2 + iX^1), \tag{3.3}$$

$$\Omega^{22} = -\frac{1}{2}(X^2 - iX^1), \tag{3.4}$$

$$\Omega^{12} = \Omega^{21} = \frac{1}{2}iX^3, \tag{3.5}$$

so that

$$\Omega^A{}_B = X^a T^{aA}{}_B. \tag{3.6}$$

Note that this differs from [17]. Now the algebraic relations satisfied by the bilinears imply the following condition given in [17]

$$V_\mu V^\mu = 0. \tag{3.7}$$

We introduce a null orthonormal basis

$$ds^2 = 2e^+e^- - \delta_{ij}e^ie^j, \tag{3.8}$$

where $e^+ = V$, and we choose the orientation

$$\epsilon^{+-1234} = 1. \tag{3.9}$$

The algebraic relations also imply [17]

$$-\frac{1}{2}X^a = V \wedge I^a, \tag{3.10}$$

where

$$I^a = \frac{1}{2}I^a_{ij}e^i \wedge e^j \tag{3.11}$$

are anti-selfdual on the 4-d base with orientation

$$\epsilon^{ijkl} = \epsilon^{+-ijkl}, \tag{3.12}$$

and obey

$$(I^a)^i{}_j(I^b)^j{}_k = \epsilon^{abc}(I^c)^i{}_k - \delta^{ab}\delta^i_k \tag{3.13}$$

where the indices have been raised with $-\delta^{ij}$. It is also shown in [17] that the Killing spinor must satisfy the projection

$$\Gamma^+\epsilon = 0. \tag{3.14}$$

3.1 Differential constraints

Employing the Killing spinor equation, one may show that the covariant derivative of V is given by

$$\nabla_\mu V_\nu = e^{\sqrt{2}\phi}V^\sigma G^+_{\sigma\mu\nu}, \tag{3.15}$$

so V is Killing and $dV = 2e^{\sqrt{2}\phi}i_V G^+$, where $i_V K$ means V contracted on the first index of the form K. For X^a we find

$$\nabla_\alpha X^a_{\mu\nu\sigma} = -g\epsilon^{abc}A^b_\alpha X^c_{\mu\nu\sigma} + e^{\sqrt{2}\phi}(G^{+\tau}{}_{\alpha\mu}X^a_{\tau\nu\sigma} + G^{+\tau}{}_{\alpha\nu}X^a_{\tau\sigma\mu} + G^{+\tau}{}_{\alpha\sigma}X^a_{\tau\mu\nu}). \tag{3.16}$$

We see that $\mathcal{L}_V X^a = 0$ if we choose the gauge $i_V A^a = 0$.

3.2 $\delta\chi = 0$

Now we turn to the analysis of (2.5). On contracting $\delta\chi = 0$ with $\bar{\epsilon}^B$ we find that

$$V^\mu\partial_\mu\phi = 0, \tag{3.17}$$

$$X^{a\,\mu\nu\sigma}G^-_{\mu\nu\sigma} = 0. \tag{3.18}$$

The duality relation for the gamma matrices is

$$\Gamma^{\mu_1\cdots\mu_n} = \frac{(-1)^{[n/2]}}{(6-n)!}\epsilon^{\mu_1\cdots\mu_n\nu_1\cdots\nu_{6-n}}\Gamma_{\nu_1\cdots\nu_{6-n}}\Gamma_7, \tag{3.19}$$

with

$$\epsilon^{012345} = 1, \tag{3.20}$$

and in our null basis,

$$\Gamma_+ = \frac{1}{\sqrt{2}}(\Gamma^0 + \Gamma^5), \tag{3.21}$$

$$\Gamma_- = \frac{1}{\sqrt{2}}(\Gamma^0 - \Gamma^5). \tag{3.22}$$

Contracting $\delta\chi = 0$ with $\bar{\epsilon}^B\Gamma_{\mu\nu}$ and employing the duality relation we find

$$V \wedge d\phi = \sqrt{2}\, i_V G^-, \tag{3.23}$$

$$G^-_{\tau\rho[\mu} X^{a\,\tau\rho}_{\nu]} = -X^a_{\mu\nu}{}^{\sigma}\partial_\sigma(e^{-\sqrt{2}\phi}). \tag{3.24}$$

Together, these imply

$$e^{\sqrt{2}\phi}G^- = (1 - \star)(-\frac{1}{\sqrt{2}}V \wedge e^- \wedge d\phi + \frac{1}{2}V \wedge K), \tag{3.25}$$

for some two form $K = \frac{1}{2}K_{ij}e^i \wedge e^j$. The anti self-duality of G^- implies that K is self dual with respect to $-\delta_{ij}$.

3.3 $\delta\lambda^{aA} = 0$

In analysing the consequences of this equation it is convenient to note the following for any product of gamma matrices A:

$$\frac{1}{2}\bar{\epsilon}^A F^a_{\mu\nu}[\Gamma^{\mu\nu}, A]\epsilon^B + ge^{-\sqrt{2}\phi}(T^{aA}{}_C\delta^B_D + T^{aB}{}_D\delta^A_C)\bar{\epsilon}^C A\epsilon^D = 0, \tag{3.26}$$

$$\frac{1}{2}\bar{\epsilon}^A F^a_{\mu\nu}\{\Gamma^{\mu\nu}, A\}\epsilon^B + ge^{-\sqrt{2}\phi}(T^{aA}{}_C\delta^B_D - T^{aB}{}_D\delta^A_C)\bar{\epsilon}^C A\epsilon^D = 0. \tag{3.27}$$

Now, with $A = \Gamma_\sigma$, we find

$$i_V F = 0, \tag{3.28}$$

$$F^{a\,\mu\nu}X^b_{\sigma\mu\nu} = 2ge^{-\sqrt{2}\phi}\delta^{ab}V_\sigma. \tag{3.29}$$

With $A = \Gamma_{\mu\nu\sigma}$ and using (3.19) we find

$$6X^b_{[\sigma\rho}{}^{\nu}F^a_{\tau]\nu} = -ge^{-\sqrt{2}\phi}\epsilon^{abc}X^c_{\sigma\rho\tau}, \tag{3.30}$$

$$V \wedge F^a + \star(V \wedge F^a) = \frac{1}{4}ge^{-\sqrt{2}\phi}X^a. \tag{3.31}$$

These imply that

$$F^a = V \wedge \omega^a_F + \tilde{F}^a - \frac{g}{4}e^{-\sqrt{2}\phi}I^a, \tag{3.32}$$

where $\omega^a_F = \omega^a_{Fi}e^i$, and $\tilde{F}^a = \frac{1}{2}\tilde{F}^a_{ij}e^i \wedge e^j$ is self dual with respect to $-\delta_{ij}$. Now one may show that

$$\mathcal{L}_V G = \mathcal{L}_V F^a = 0, \tag{3.33}$$

and in the gauge chosen above, also that $\mathcal{L}_V A^a = 0$.

3.4 Sufficient conditions for supersymmetry

Now we show that the necessary conditions for supersymmetry we have derived are (when supplemented by further projections from the gauge multiplet) also sufficient. We begin by analysing $\delta\chi$. In the basis chosen above, given the projection (3.14), and noting that Γ^+ and Γ^- do not anticommute, it is immediate that if G^- is given by (3.25), then $\delta\chi$ reduces to

$$\delta\chi^A = i\left(-\frac{1}{2\sqrt{2}}\Gamma^i\partial_i\phi - \frac{1}{12\sqrt{2}}\epsilon_{ijkl}\Gamma^{ijk}\partial^l\phi\right)\epsilon^A. \tag{3.34}$$

Now we note that on the base,

$$\Gamma^{i_1 \ldots i_n} = \frac{(-1)^{[n/2]}}{(4-n)!} \epsilon^{i_1 \ldots i_n j_i \ldots j_{4-n}} \Gamma_{j_1 \ldots j_{4-n}} \Gamma_\star, \tag{3.35}$$

where $\Gamma_\star \equiv \Gamma_{1234}$. Since $\Gamma_7 \epsilon = -\epsilon$ and $\Gamma^+ \epsilon = 0$ we have that $\Gamma_\star \epsilon = \epsilon$. Thus $\delta\chi = 0$.

Next we turn to $\delta\lambda^a$. If F^a is given by (3.32) then again because of the projection (3.14), the $V \wedge \omega_F$ term gives zero. The self duality of $\tilde{F}^a$ with respect to $-\delta_{ij}$ implies that $\tilde{F}^a_{ij} \Gamma^{ij} \epsilon = 0$. Thus we find that

$$\delta\lambda^{aA} = \frac{g e^{-\phi/\sqrt{2}}}{\sqrt{2}} \Big(\frac{1}{8} I^a_{ij} \Gamma^{ij} \delta^A_B + T^{aA}{}_B \Big) \epsilon^B. \tag{3.36}$$

At first it might appear that the vanishing of $\delta\lambda^a$ requires three further projections on ϵ thus breaking all supersymmetry. However we note that

$$\Big[\Big(\frac{1}{8} I^a_{ij} \Gamma^{ij} + T^a \Big), \Big(\frac{1}{8} I^b_{kl} \Gamma^{kl} + T^b \Big) \Big]^A_B = \epsilon^{abc} \Big(\frac{1}{8} I^c_{ij} \Gamma^{ij} \delta^A_B + T^{cA}{}_B \Big), \tag{3.37}$$

so any two projections imply the third and generically one supersymmetry is left unbroken. In the $U(1)$ theory we have only one gaugino so only one further projection, and thus generically two supersymmetries are unbroken.

Finally we turn to the Killing spinor equation. All terms involving a Γ^- vanish because (3.15) is equivalent to

$$\omega_{\mu\nu -} = e^{\sqrt{2}\phi} G^{(+)}_{\mu\nu -}. \tag{3.38}$$

Then all further terms involving a Γ^+ vanish because of (3.14). Therefore we find that

$$\delta\psi_\mu = (\partial_\mu + \frac{1}{4} \omega_{\mu ij} \Gamma^{ij} + g A^a_\mu T^a - \frac{e^{\sqrt{2}\phi}}{4} G^{(+)}_{\mu ij} \Gamma^{ij}) \epsilon. \tag{3.39}$$

Now the terms involving the parts of $\omega_{\mu ij}$ and $G_{\mu ij}$ which are self dual in the indices i, j vanish as above. To analyse the anti-self dual part we note that (3.16) implies that

$$\nabla_\alpha I^a_{ij} + g \epsilon^{abc} A^b_\alpha I^c_{ij} = e^{\sqrt{2}\phi} (G^{(+)k}_{\alpha i} I^a_{kj} - G^{(+)k}_{\alpha j} I^a_{ki}), \tag{3.40}$$

and, if we choose the basis so that the components of the I^a are constants, this becomes

$$(\omega_{\alpha ij} - e^{\sqrt{2}\phi} G^{(+)}_{\alpha ij})^{\tilde{-}} = \frac{g}{2} A^a_\alpha I^a_{ij}, \tag{3.41}$$

where $\tilde{-}$ denotes the anti-self dual projection in i, j. Hence the variation of the gravitino reduces to

$$\delta\psi_\mu = \Big(\partial_\mu + g A^a_\mu \Big(\frac{1}{8} I^a_{ij} \Gamma^{ij} + T^a \Big) \Big) \epsilon$$
$$= \partial_\mu \epsilon, \tag{3.42}$$

if ϵ obeys the projections required for the vanishing of $\delta\lambda$. Thus in this basis, given the algebraic and differential constraints on V and X, and the form of the fields given above, the Killing spinor equation is satisfied by any constant spinor satisfying the requisite projections. Thus we have derived necessary and sufficient conditions for supersymmetry.

4. The field equations

In this section we will impose the field equations on our general ansatz. We introduce the local coordinates of [17]:

$$ds^2 = 2H^{-1}(du + \beta_m dx^m)\Big(dv + \omega_n dx^n + \frac{\mathcal{F}}{2}(du + \beta_n dx^n)\Big) - Hh_{mn}dx^m dx^n, \qquad (4.1)$$

with

$$\begin{aligned} e^+ &= H^{-1}(du + \beta_m dx^m), \\ e^- &= dv + \omega_m dx^m + \frac{\mathcal{F}\mathcal{H}}{2}e^+. \end{aligned} \qquad (4.2)$$

As vectors,

$$e^+ = \frac{\partial}{\partial v}, \qquad (4.3)$$

$$e^- = H\Big(\frac{\partial}{\partial u} - \frac{\mathcal{F}}{2}\frac{\partial}{\partial v}\Big), \qquad (4.4)$$

and H, $\mathcal{F}$, ω, β and h_{mn} depend on u and x but not on v, which is the affine parameter along the null geodesics to which the Killing vector e^+ is tangent.

In what follows we will employ some further notation of [17]. Specifically, let Φ be a form defined on the base with

$$\Phi = \frac{1}{p!}\Phi_{i_1...i_p}(u,x)dx^{i_1} \wedge ... \wedge dx^{i_p}, \qquad (4.5)$$

and let

$$\tilde{d}\Phi = \frac{1}{(p+1)!}(p+1)\frac{\partial}{\partial x^{[q}}\Phi_{i_1...i_p]}dx^q \wedge dx^{i_1} \wedge ... \wedge dx^{i_p}, \qquad (4.6)$$

and define the operator $\mathcal{D}$ as

$$\mathcal{D}\Phi = \tilde{d}\Phi - \beta \wedge \dot{\Phi}, \qquad (4.7)$$

where $\dot{\Phi}$ denotes the Lie derivative with respect to $\frac{\partial}{\partial u}$. Then we have

$$d\Phi = \mathcal{D}\Phi + He^+ \wedge \dot{\Phi}. \qquad (4.8)$$

Further note that

$$de^+ = H^{-1}\mathcal{D}\beta + e^+ \wedge (H^{-1}\mathcal{D}H + \dot{\beta}), \qquad (4.9)$$

$$de^- = \mathcal{D}\omega + \frac{\mathcal{F}}{2}\mathcal{D}\beta + He^+ \wedge (\dot{\omega} + \frac{\mathcal{F}}{2}\dot{\beta} - \frac{1}{2}\mathcal{D}\mathcal{F}). \qquad (4.10)$$

We define $J^a = H^{-1}I^a$. They obey

$$(J^a)^i{}_j(J^b)^j{}_k = -\epsilon^{abc}(J^c)^i{}_k - \delta^{ab}\delta^j{}_k, \qquad (4.11)$$

where, here and henceforth, indices on the base are raised with h^{mn}. If we define

$$A^a = A^a_+(u,x)e^+ + \tilde{A}^a, \quad \tilde{A}^a = A^a_i(u,x)e^i, \qquad (4.12)$$

then equation (3.16) implies that

$$\tilde{d}J^a + g\epsilon^{abc}\tilde{A}^b \wedge J^c = \partial_u(\beta \wedge J^a).\qquad(4.13)$$

Antisymmetrising the kth component of (3.40), employing the expression for G^+ given below, and comparing with (4.13), we see that $\beta \wedge \partial_u(HJ^a) = 0$.

We still have the freedom to make a v-independent gauge transformation on the 1-form potential, while preserving the condition $i_V A^a = 0$. Since A_+^a is independent of v, we may exploit this freedom to set $A_+^a = 0$ which we do in what follows. Also we note that (3.17) implies that $\phi = \phi(u,x)$.

4.1 The three form

Equations (3.25), (3.38) and the $+$ component of (3.40) imply that

$$e^{\sqrt{2}\phi}G = \frac{1}{2}\star_4(\mathcal{D}H + H\dot{\beta} - \sqrt{2}H\mathcal{D}\phi) + e^+ \wedge \left(-H\psi - \frac{1}{2}(\mathcal{D}\omega)^- + K\right)$$
$$+ e^- \wedge \frac{\mathcal{D}\beta}{2H} - \frac{1}{2}e^+ \wedge e^- \wedge (H^{-1}\mathcal{D}H + \dot{\beta} + \sqrt{2}\mathcal{D}\phi),\qquad(4.14)$$

where

$$\psi = \frac{H}{16}\epsilon^{abc}J^{aij}j_{ij}^b J^c,\qquad(4.15)$$

and also

$$\mathcal{D}\beta = \star_4\mathcal{D}\beta,\qquad(4.16)$$

with $\star_4$ the Hodge dual on the base with metric h_{mn}. If we write $\delta\psi_\mu = D_\mu\epsilon$, $\delta\chi = \Delta_G\epsilon$, then we have the integrability condition

$$i\Gamma^\mu[D_\mu, \Delta_G]\epsilon = \frac{1}{\sqrt{2}}\Big(\nabla^2\phi - \frac{1}{3\sqrt{2}}e^{2\sqrt{2}\phi}G_{\mu\nu\sigma}G^{\mu\nu\sigma} + \frac{1}{2\sqrt{2}}e^{\sqrt{2}\phi}F_{\mu\nu}^a F^{a\mu\nu}$$
$$- \frac{ng^2}{4\sqrt{2}}e^{-\sqrt{2}\phi}\Big)\epsilon + \frac{1}{4}e^{-\sqrt{2}\phi}(\star d\star(e^{2\sqrt{2}\phi}G))_{\mu\nu}\Gamma^{\mu\nu}\epsilon$$
$$+ \frac{1}{48}e^{\sqrt{2}\phi}(dG - F^a \wedge F^a)_{\mu\nu\sigma\tau}\Gamma^{\mu\nu\sigma\tau}\epsilon - \frac{i}{3}G_{\mu\nu\sigma}\Gamma^{\mu\nu\sigma}\delta\chi$$
$$- \Big(\frac{1}{2\sqrt{2}}e^{\phi/\sqrt{2}}F_{\mu\nu}^a\Gamma^{\mu\nu} + \frac{g}{\sqrt{2}}e^{-\phi/\sqrt{2}}T^a\Big)\delta\lambda^a = 0.\qquad(4.17)$$

Thus the existence of a Killing spinor means that if we impose the three form Bianchi identities and field equations, the dilaton field equation is automatically satisfied. Defining

$$\mathcal{G} = \frac{1}{2H}((\mathcal{D}\omega)^+ + \frac{1}{2}\mathcal{F}\mathcal{D}\beta),\qquad(4.18)$$

the $+ij$ components of the three form field equations and the Bianchi identities give

$$\mathcal{D}(H^{-1}e^{\sqrt{2}\phi}(K - H\mathcal{G} - H\psi)) + \frac{1}{2}\partial_u \star_4 (\mathcal{D}(He^{\sqrt{2}\phi}) + e^{\sqrt{2}\phi}H\dot{\beta})$$
$$- H^{-1}e^{\sqrt{2}\phi}\dot{\beta} \wedge (K - H\mathcal{G} - H\psi) = 0\qquad(4.19)$$

and

$$-\mathcal{D}(H^{-1}e^{-\sqrt{2}\phi}(K + H\mathcal{G} - H\psi)) + \frac{1}{2}\partial_u \star_4 (\mathcal{D}(He^{-\sqrt{2}\phi}) + e^{-\sqrt{2}\phi}H\dot{\beta})$$

$$+ H^{-1}e^{-\sqrt{2}\phi}\dot{\beta} \wedge (K + H\mathcal{G} - H\psi) = 2H^{-1}\omega_F^a \wedge (\tilde{F}^a - \frac{g}{4}He^{-\sqrt{2}\phi}J^a), \qquad (4.20)$$

while the ijk components are

$$\frac{1}{2}e^{-\sqrt{2}\phi}\mathcal{D} \star_4 \left(\mathcal{D}(e^{\sqrt{2}\phi}H) + e^{\sqrt{2}\phi}H\dot{\beta}\right) = H^{-1}(K - H\mathcal{G}) \wedge \mathcal{D}\beta, \qquad (4.21)$$

$$\frac{1}{2}e^{\sqrt{2}\phi}\mathcal{D} \star_4 \left(\mathcal{D}(e^{-\sqrt{2}\phi}H) + e^{-\sqrt{2}\phi}H\dot{\beta}\right) + H^{-1}(K + H\mathcal{G}) \wedge \mathcal{D}\beta$$

$$= e^{\sqrt{2}\phi}\tilde{F}^a \wedge \tilde{F}^a + \frac{H^2}{16}g^2 e^{-\sqrt{2}\phi}J^a \wedge J^a. \qquad (4.22)$$

4.2 The two forms

Writing $\delta\lambda^a = \Delta_F^a\epsilon$, we obtain the following integrability condition

$$\sqrt{2}\Gamma^\mu[D_\mu, \Delta_F^a]\epsilon = -\frac{1}{6}e^{\phi/\sqrt{2}}(dF^a + g\epsilon^{abc}A^b \wedge F^c)_{\mu\nu\sigma}\Gamma^{\mu\nu\sigma}\epsilon$$

$$- e^{-\phi/\sqrt{2}}P_\mu^a\Gamma^\mu\epsilon - \Gamma^\mu\partial_\mu\phi\delta\lambda^a - ie^{\phi/\sqrt{2}}F_{\mu\nu}^a\Gamma^{\mu\nu}\delta\chi$$

$$+ \sqrt{2}i[\Delta_G, \Delta_F^a]\epsilon - \sqrt{2}g\Gamma^\mu\epsilon^{abc}A_\mu^b\delta\lambda^c, \qquad (4.23)$$

with

$$P_\mu^a = \left(\star \left[d \star (e^{\sqrt{2}\phi}F^a) + 2e^{2\sqrt{2}\phi} \star G \wedge F^a + ge^{\sqrt{2}\phi}\epsilon^{abc}A^b \wedge \star F^c\right]\right)_\mu. \qquad (4.24)$$

The two form field equations are $P_\mu^a = 0$. If we impose the Bianchi identities we have $\Gamma^\mu P_\mu^a\epsilon = 0$. Acting with $\bar{\epsilon}$ and $\Gamma^\nu P_\nu^a$ we find $P_-^a = P_\mu^a P^{a\mu} = 0$. Hence the existence of a Killing spinor together with the Bianchi identities implies that all except the $+$ component of the field equations are automatically satisfied. Imposing $F^a = dA^a + \frac{g}{2}\epsilon^{abc}A^b \wedge A^c$, we find that

$$\tilde{F}^a - \frac{g}{4}e^{-\sqrt{2}\phi}HJ^a = \mathcal{D}\tilde{A}^a + \frac{g}{2}\epsilon^{abc}\tilde{A}^b \wedge \tilde{A}^c, \qquad (4.25)$$

$$\omega_F^a = H\dot{\tilde{A}}^a. \qquad (4.26)$$

We may invert the $\alpha = k$ components of equation (3.40) to solve for A and obtain

$$A_i^a = \frac{1}{8g}\left[- \epsilon^{abc}J^{bjk}\tilde{\nabla}_i J_{jk}^c - \beta_i\epsilon^{abc}J^{bjk}\dot{j}_{ij}^c + 4\dot{\beta}_j J^{aj}{}_i + 2\beta_j J_{ki}^a\dot{h}^{jk}\right], \qquad (4.27)$$

where $\tilde{\nabla}$ denotes the Levi-Civita connection on the base with metric h_{mn}. The $+$ component of the field equation gives an equation for ω_F^a and reads

$$\mathcal{D}(H \star_4 \omega_F^a) = -2\sqrt{2}\mathcal{D}\phi \wedge (H \star_4 \omega_F^a) + 2\tilde{F}^a \wedge (K - H\mathcal{G})$$

$$- \frac{g}{2}He^{-\sqrt{2}\phi}J^a \wedge (H\psi + (\mathcal{D}\omega)^-) - gH\epsilon^{abc}\tilde{A}^b \wedge \star_4\omega_F^c. \qquad (4.28)$$

$$- 10 -$$

4.3 The Einstein equations

It may be shown with considerable effort that the integrability condition for the Killing spinor equation is

$$\Gamma^\nu[D_\mu, D_\nu]\epsilon = \frac{1}{2}E_{\mu\nu}\Gamma^\nu\epsilon + \frac{e^{\sqrt{2}\phi}}{96}(dG - F^a \wedge F^a)_{\alpha\beta\gamma\delta}\Gamma^{\alpha\beta\gamma\delta}\Gamma_\mu\epsilon$$

$$+ \frac{e^{-\sqrt{2}\phi}}{8}(\star d \star (e^{2\sqrt{2}\phi}G))_{\alpha\beta}\Gamma^{\alpha\beta}\Gamma_\mu\epsilon$$

$$+ \sqrt{2}e^{\phi/\sqrt{2}}F_{\mu\nu}^a\Gamma^\nu\delta\lambda^a - i(\sqrt{2}\partial_\mu\phi - \frac{1}{12}G_{\alpha\beta\gamma}^-\Gamma^{\alpha\beta\gamma}\Gamma_\mu)\delta\chi$$

$$- \frac{1}{2\sqrt{2}}\Gamma_\mu\left(\frac{e^{\phi/\sqrt{2}}}{2}F_{\alpha\beta}^a\Gamma^{\alpha\beta} + ge^{-\phi/\sqrt{2}}T^a\right)\delta\lambda^a, \tag{4.29}$$

where

$$E_{\mu\nu} = -R_{\mu\nu} + 2\partial_\mu\phi\partial_\nu\phi + e^{2\sqrt{2}\phi}(G_{\mu\sigma\lambda}G_\nu{}^{\sigma\lambda} - \frac{1}{6}g_{\mu\nu}G_{\alpha\beta\gamma}G^{\alpha\beta\gamma})$$

$$+ 2e^{\sqrt{2}\phi}(-F_{\mu\alpha}^aF_\nu^{a\,\alpha} + \frac{1}{8}g_{\mu\nu}F_{\alpha\beta}^aF^{a\alpha\beta}) - \frac{ng^2}{8}e^{-\sqrt{2}\phi}g_{\mu\nu}, \tag{4.30}$$

and the Einstein equations are

$$E_{\mu\nu} = 0. \tag{4.31}$$

Given the vanishing of the supersymmetry variations of χ and λ^a, and that G satisfies its field equation and Bianchi identity, we see upon acting on (4.29) with $\bar{\epsilon}$ and $E_{\mu\sigma}\Gamma^\sigma$ that all except the $++$ component of the Einstein equations are implied by the integrability of the Killing spinor equation. The $++$ component is

$$\star_4\mathcal{D}(\star_4[\dot{\omega} + \frac{1}{2}\mathcal{F}\dot{\beta} - \frac{1}{2}\mathcal{D}\mathcal{F}]) = \frac{1}{2}Hh^{mn}\partial_u^2(Hh_{mn}) + \frac{1}{4}\partial_u(Hh^{mn})\partial_u(Hh_{mn})$$

$$- 2\dot{\beta}_m(\dot{\omega} + \frac{1}{2}\mathcal{F}\dot{\beta} - \frac{1}{2}\mathcal{D}\mathcal{F})^m - \frac{1}{2}H^{-2}(\mathcal{D}\omega + \frac{1}{2}\mathcal{F}\mathcal{D}\beta)^2$$

$$+ 2H^{-2}(K + H\psi + \frac{1}{2}(\mathcal{D}\omega)^-)^2 + 2H^2\dot{\phi}^2 + 2H^{-1}e^{\sqrt{2}\phi}\omega_F^{ai}\omega_{Fi}^a \tag{4.32}$$

where for a two form M,

$$\mathcal{M}^2 = \frac{1}{2}M_{ij}M^{ij}. \tag{4.33}$$

5. Fluxes and intrinsic torsion

The geometrical structure we have studied so far is associated to a chiral spinor and is given by $SU(2) \ltimes \mathbb{R}^4$. $SU(2)$ corresponds to rotations in the base, while $\mathbb{R}^4$ to null rotations that leave the Killing spinor invariant. The objects that define such a structure are the Killing vector K and the triplet of anti-selfdual forms I^a. The presence of fluxes in the Killing equation implies that such objects are not covariantly constant with respect to the Levi-Civita connection, but are covariantly constant with respect to a connection with torsion. One can then classify the various inequivalent spacetimes by studying their intrinsic torsion. The intrinsic torsion can be roughly described as the obstruction to finding a torsion free

connection on the spacetime. Given a pair of connections, their different is always a tensor field $\alpha^\lambda_{\mu\nu}$, with one covariant and two contravariant indices. The difference of their associated torsion tensors is then given by $\alpha^\lambda_{\nu\mu} - \alpha^\lambda_{\mu\nu}$. It is clear then that it is possible to obtain a new connection with zero torsion if and only if the original torsion field could be written as $\alpha^\lambda_{\mu\nu} - \alpha^\lambda_{\nu\mu}$ for some tensor field α. Given a spacetime with a torsion tensor $T^\lambda_{\mu\nu} = T^\lambda_{[\mu\nu]}$, its intrinsic torsion is defined as the projection of T on the quotient space obtained via this subtraction procedure.

The gaugino transformation law implies a set of projections on ϵ given by (3.36). Plugging these into (2.4) one can rewrite this as

$$\delta\psi^A_\mu = \nabla^T_\mu \epsilon^A = \left[\partial_\mu + \frac{1}{4}\left(\omega_{\mu\hat\alpha\hat\beta} + T_{\mu\hat\alpha\hat\beta}\right)\Gamma^{\hat\alpha\hat\beta}\right]\epsilon^A, \tag{5.1}$$

where $\hat\alpha$, $\hat\beta$ are flat $6D$ indices, and $T_{\lambda\mu_1\mu_2}$ is a tensor given by

$$T_{\lambda\mu_1\mu_2} = -\left(e^{\sqrt{2}\Phi}G^+_{\lambda\mu_1\mu_2} + \frac{g}{2}A^a_\lambda I^a_{ij}\delta^i_{\mu_1}\delta^j_{\mu_2}\right). \tag{5.2}$$

λ is the index associated to differentiation, while μ_1, μ_2 are cotangent space indices associated to the group action of $SO(6)$ on tensors. Eq.(5.2) is clearly antisymmetric in the indices μ_1, μ_2, but in general it is not with respect to λ and μ_2. This latter symmetry is that required in order to interpret T as a torsion tensor. Now, notice that equations (3.15), (3.40) can be rewritten as

$$\nabla^T_\mu V_\nu = 0,$$
$$\nabla^T_\mu I^a_{ij} = 0. \tag{5.3}$$

This tells us that there exists a suitable connection such that the $SU(2) \ltimes \mathbb{R}^4$ structure can be seen as covariantly constant, as in the case without fluxes. The overall effect of fluxes is that of deforming the geometry, while keeping the same geometric structure. When (5.2) is antisymmetric in the indices λ, μ_2 then we can think of it as a torsion tensor associated to the Levi-Civita connection. If there is a part symmetric in λ, μ_2 then instead one has to consider a connection which is more general than the Levi-Civita one.

In principle one can study in detail the intrinsic torsion in six dimensions. However, as we are going to see later, for most of the applications one is mostly interested in fully understanding the geometry on the four-dimensional base. Therefore we are going to restrict ourselves to systematically describe the geometry on the base. All the information that is not included in such geometry is encoded in $\dot{\hat{A}}^a$ and in the components of G^+ laying along u and v directions. These basically correspond to derivatives of H and of the twisting parameters β, ω appearing in the mertic (4.1).

On the base there is an $Sp(1)$ structure with torsion, see below for a definition. In order to calculate it project equation (3.40) on the base and get

$$\tilde\nabla_i J^a_{jk} + g\epsilon^{abc}A^b_i J^c_{jk} - \beta_i \dot J^a_{jk} - \dot\beta_{[j}J^a_{k]i} + h_{i[j}\dot\beta^m J^a_{k]m} = 0. \tag{5.4}$$

Where indices are raised with the metric h^{ij}. This corresponds to the vanishing of a covariant derivative with a tensorial part given by

$$\tilde T_{ijk} = \frac{g}{2}A^a_i J^a_{jk} + \beta_i \tilde e^{\hat a}_j \dot{\tilde e}_{\hat a k} - \left(h_{i[j}\dot\beta_{k]}\right). \tag{5.5}$$

Again there is explicit antisymmetry with respect to the indices j, k, but not with respect to i and k, and therefore the same remarks made for eq.(5.2) apply. In four dimensions a rather general class of manifolds such that the tensor (5.5) can be directly seen as a torsion tensor is given by Hyper Kähler manifolds with torsion (HKT), which are those such that (5.5) is completely antisymmetric.

An $Sp(1)$ structure can be shown to be equivalent to an $SU(2)$ one. In our case the $SU(2)$ structure can be obtained once we choose one of the three J^a to be a complex structure. Choose J^1 for example and define

$$J = J^1,$$
$$\Omega = J^2 + iJ^3. \tag{5.6}$$

It is not difficult to show that they define an $SU(2)$ structure, that satisfies the defining equations

$$J \wedge \Omega = 0,$$
$$\Omega \wedge \overline{\Omega} = 2J \wedge J. \tag{5.7}$$

$SU(2)$ structures are completely understood. Their intrinsic torsion can be decomposed into invariant representations, called modules. In our case there are three such modules, given by

$$\mathcal{W}_2 = \frac{1}{4} \left(\Omega \lrcorner d\Omega + \overline{\Omega} \lrcorner d\overline{\Omega} \right),$$
$$\mathcal{W}_4 = J \lrcorner dJ,$$
$$\mathcal{W}_5 = \frac{1}{4} \left(\Omega \lrcorner d\overline{\Omega} + \overline{\Omega} \lrcorner d\Omega \right),$$

where we define the contraction $\lrcorner$ as $(\omega_2 \lrcorner \omega_3)_k := 1/2\omega_2^{ij} \, \omega_{3\,ijk}$. Now, calculate dJ, $d\Omega$ using (4.14) and $\beta \wedge \partial_u(HJ) = 0$. Equivalently one can antisymmetrize (5.4) and obtain the same, as a check of consistency. The result is

$$\mathcal{W}_2 = \frac{g}{2} \left(A^2 \, J^2 - A^3 \, J^3 \right),$$
$$\mathcal{W}_4 = -g \left(A^2 \, J^2 + A^3 \, J^3 \right) + H\partial_u(H^{-1}\beta),$$
$$\mathcal{W}_5 = -\frac{g}{2}(A^2 J^2 + A^3 J^3) - gA^1 J^1 + H\partial_u(H^{-1}\beta).$$

When $\mathcal{W}_2 = 0$ the structure is integrable, and this corresponds to $A^2 \lrcorner J^2 = A^3 \lrcorner J^3$. When both $\mathcal{W}_2$ and $\mathcal{W}_4$ are zero instead the manifold is Kähler. This corresponds, for $\partial_u(H^{-1}\beta) = 0$, to $A^2 = 0 = A^3$ that is, the $U(1)$ theory. This case will be studied in detail in the next section.

6. Supersymmetric solutions

When either $\beta = 0$ or the full system is independent of u, the general problem simplifies considerably. We then have

$$\tilde{\nabla}_i J^a_{jk} = -g\epsilon^{abc} A^b_i J^c_{jk},$$
$$A^a_i = -\frac{1}{8g}\epsilon^{abc} J^{bij} \tilde{\nabla}_i J^c_{jk}. \tag{6.1}$$

Note that for the $U(1)$ theory, this implies that the base is Kähler, since then one of the Js is covariantly constant on the base. Now using (6.1) and

$$\tilde{F}^a - \frac{g}{4}e^{-\sqrt{2}\phi}HJ^a = \tilde{d}\tilde{A}^a + \frac{g}{2}\epsilon^{abc}\tilde{A}^b \wedge \tilde{A}^c, \tag{6.2}$$

we find that

$$(\tilde{F}^a - \frac{g}{4}e^{-\sqrt{2}\phi}HJ^a)_{ij} = -\frac{1}{2g}J^{amn}R_{mnij} \equiv -\frac{1}{g}\mathfrak{R}^a_{ij}, \tag{6.3}$$

with R_{ijkl} the Riemann tensor on the base with metric h_{mn}. Contracting with J^a we find

$$e^{-\sqrt{2}\phi}H = \frac{1}{ng^2}\mathfrak{R}^a_{ij}J^{aij} \equiv \frac{1}{g^2}\mathfrak{R}. \tag{6.4}$$

As we shall see below, $\mathfrak{R}$ is proportional to the scalar curvature of the base, with positive constant of proportionality. Thus, since we are taking H positive, (this amounts to the our choice of the signature of spacetime) the base must have positive scalar curvature. If we in fact assume that $\beta = 0$ (that is, we seek what we shall refer to as non-twisting solutions), then on employing (6.3) and (6.4), we find that (4.21) and (4.22) become

$$\tilde{\nabla}^2(e^{2\sqrt{2}\phi}\mathfrak{R}) = 0, \tag{6.5}$$

$$\tilde{\nabla}^2\mathfrak{R} = \frac{n}{2}\mathfrak{R}^2 - \mathfrak{R}^a_{ij}\mathfrak{R}^{aij}. \tag{6.6}$$

Thus we see that we must choose a (possibly u-dependent) base satisfying (6.6). Then we choose a harmonic function, f, on the base, and we have

$$e^{2\sqrt{2}\phi} = \frac{f}{\mathfrak{R}}, \tag{6.7}$$

$$H = \frac{1}{g^2}\sqrt{f\mathfrak{R}}. \tag{6.8}$$

We now employ the gauge freedom present in our choice of coordinates to set $\omega = 0$. Then $\mathcal{G} = 0$ also, and on rescaling ω^a_F according to

$$\omega^a_F = e^{-\sqrt{2}\phi}\tilde{\omega}^a_F, \tag{6.9}$$

and defining the two form Y as

$$Y = K - H\psi, \tag{6.10}$$

equations (4.19), (4.20) and (4.28) reduce to the coupled system

$$\tilde{d}\left(\frac{1}{\mathfrak{R}}\star_4 Y\right) = -\frac{1}{2g^4}\partial_u(\star_4\tilde{d}f), \tag{6.11}$$

$$\tilde{d}\left(\frac{1}{f}Y\right) = \frac{1}{2g^4}\partial_u(\star_4\tilde{d}\mathfrak{R}) + \frac{2}{gf}\tilde{\omega}^a_F \wedge \mathfrak{R}^a, \tag{6.12}$$

$$\tilde{d}(\star_4 f\tilde{\omega}^a_F) = -\frac{2gf}{\mathfrak{R}}\mathfrak{R}^a \wedge \star_4 Y + g\epsilon^{abc}\tilde{A}^b \wedge f \star_4 \tilde{\omega}^c_F. \tag{6.13}$$

The $++$ component of the Einstein equation is

$$\frac{1}{2}\tilde{\nabla}^2\mathcal{F} = \frac{1}{g^4}\left[\mathfrak{R}^2\left(\partial_u\sqrt{\frac{f}{\mathfrak{R}}}\right)^2 + \frac{1}{2}\sqrt{f\mathfrak{R}}h^{mn}\partial_u^2(\sqrt{f\mathfrak{R}}h_{mn})\right.$$
$$\left. + \frac{1}{4}\partial_u(\sqrt{f\mathfrak{R}}h^{mn})\partial_u(\sqrt{f\mathfrak{R}}h_{mn})\right] + \frac{2g^4}{f\mathfrak{R}}Y^2 + \frac{2g^2}{f}(\tilde{\omega}^a_F)^2. \tag{6.14}$$

We will now consider in turn the $U(1)$ and $SU(2)$ theories.

6.1 Non-twisting solutions of the $U(1)$ theory

We take the $U(1)$ generator to be T^1. Then the base is Kähler with gauge-invariant Kähler form J^1 and Ricci form $\mathfrak{R}^1$. The Ricci form obeys

$$\mathfrak{R}^1_{ij} = R_{ik} J^{1k}{}_j, \tag{6.15}$$

where R_{ij} is the Ricci tensor of the base. Thus we have $R = \mathfrak{R} = J^{1ij}\mathfrak{R}^1_{ij}$, with R the Ricci scalar of the base. Equation (6.6) becomes

$$\tilde{\nabla}^2 R = \frac{1}{2}R^2 - R^{ij}R_{ij}. \tag{6.16}$$

6.1.1 Dyonic string

Let us take the base to be of the form

$$ds^2 = a^{-2}dr^2 + \frac{r^2}{4}((\sigma_R^1)^2 + (\sigma_R^2)^2) + a^2\frac{r^2}{4}(\sigma_R^3)^2, \tag{6.17}$$

where the left-invariant 1-forms σ_R^a obey $d\sigma_R^a = 1/2\epsilon^{abc}\sigma_R^b \wedge \sigma_R^c$. Taking an orthonormal basis to be given by

$$e^1 = \frac{r}{2}\sigma_R^1, \quad e^2 = \frac{r}{2}\sigma_R^2, \quad e^3 = a\frac{r}{2}\sigma_R^3, \quad e^4 = a^{-1}dr, \tag{6.18}$$

the Kähler form is $J^1 = e^1 \wedge e^2 - e^3 \wedge e^4$, and the vierbein components of the Ricci tensor are

$$R_{11} = R_{22} = \frac{4}{r^2}(1 - a^2), \tag{6.19}$$

$$R_{33} = R_{44} = 0. \tag{6.20}$$

Thus the base has everywhere positive scalar curvature if $a < 1$. Now we may recover the dyonic string solutions of [7]. Let us take the full solution to be independent of u, and take $K = \mathcal{F} = 0$. Then choosing

$$f = \frac{g^4}{8(1 - a^2)}\left(Q_1 + \frac{Q_2}{r^2}\right), \tag{6.21}$$

the full solution is completely determined; it is

$$ds^2 = \frac{r^2}{\sqrt{r^2Q_1 + Q_2}}\eta_{\mu\nu}dx^\mu dx^\nu$$
$$- \frac{1}{r}\sqrt{Q_1 + \frac{Q_2}{r^2}}\left(a^{-2}dr^2 + \frac{r^2}{4}[(\sigma_R^1)^2 + (\sigma_R^2)^2 + a^2(\sigma_R^3)^2]\right), \tag{6.22}$$

$$F = -\frac{1}{g}(1 - a^2)\sigma_R^1 \wedge \sigma_R^2, \quad e^{2\sqrt{2}\phi} = \frac{g^4(r^2Q_1 + Q_2)}{64(1 - a^2)^2}, \tag{6.23}$$

and G is determined by H and ϕ.

6.1.2 Base a product of two manifolds

Next we take the base to be a product of two real oriented Riemannian manifolds, $\mathcal{B} = \mathcal{M}_1 \times \mathcal{M}_2$. We take J^1 to be $Vol_1 - Vol_2$, where Vol_i the volume form of $\mathcal{M}_i$. Then with R_i the scalar curvature of $\mathcal{M}_i$, (6.16) gives

$$\tilde{\nabla}^2(R_1 + R_2) = R_1 R_2. \tag{6.24}$$

The Salam-Sezgin model [3] is an example of such a solution with $\mathcal{M}_1 = \mathbb{R}^2$, $\mathcal{M}_2 = S^2$. More generally, $\mathbb{R}^2 \times \mathcal{M}_2$ where $\mathcal{M}_2$ has everywhere positive harmonic scalar curvature is also an allowed solution. However we will now constuct a u-dependent generalisation of the Salam-Sezgin model, by allowing the radius of the S^2 to depend on u. We will find that the four dimensional part of the metric takes the form of a pp-wave. We take the base to be of the form

$$ds_4^2 = dx^2 + dy^2 + t^2(u)(d\theta^2 + \sin^2\theta d\phi^2). \tag{6.25}$$

We take $H = 1$, and so we have

$$R = \frac{2}{t^2}, \tag{6.26}$$

$$e^{\sqrt{2}\phi} = \frac{g^2 t^2}{2}. \tag{6.27}$$

Defining

$$e^1 = dx, \quad e^2 = dy, \quad e^3 = td\theta, \quad e^4 = t\sin\theta d\phi, \tag{6.28}$$

we choose the J^i to be

$$J^1 = e^1 \wedge e^2 - e^3 \wedge e^4, \quad J^2 = e^1 \wedge e^4 - e^2 \wedge e^3, \quad J^3 = e^1 \wedge e^3 + e^2 \wedge e^4, \tag{6.29}$$

so $\psi = 0$, $\dot{\tilde{A}} = 0$ and

$$\tilde{F} - \frac{g}{4}e^{-\sqrt{2}\phi}J^1 = \frac{1}{gt^2}e^3 \wedge e^4. \tag{6.30}$$

We choose $K = 0$, so that $G = 0$, and the Einstein equation reduces to

$$\tilde{\nabla}^2 \mathcal{F} = \frac{8\dot{t}^2}{t^2} + \frac{4\ddot{t}}{t}. \tag{6.31}$$

This is solved by

$$\mathcal{F} = \left(2\frac{\dot{t}^2}{t^2} + \frac{\ddot{t}}{t}\right)\mathbf{x}^2. \tag{6.32}$$

If we choose $t = \cosh u$, the full metric is given by

$$ds^2 = 2du(dv + \frac{1}{2}(1 + 2\tanh^2 u)\mathbf{x}^2 du) - d\mathbf{x}_2^2 - \cosh^2 u d\Omega_2^2. \tag{6.33}$$

With this choice of t, we have a sort of dynamical supersymmetric compactification; the spacetime is effectively four dimensional for small $|u|$, and decompactifies for large $|u|$.

6.1.3 Cahen-Wallach$_4 \times S^2$

We conclude our discussion of non-twisting solutions of the $U(1)$ theory by presenting the symmetric space $CW_4 \times S^2$ as a solution which preserves one quarter supersymmetry, as well as maximal four dimensional symmetry (this was not obtained by the authors of [9] as they only looked for solutions with four dimensional Poincaré, de Sitter or Anti de Sitter symmetry). Specifically, we take the base to be $\mathbb{R}^2 \times S^2$, with u-independent metric

$$ds^2 = d\mathbf{x}_2^2 + a^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{6.34}$$

With the obvious choice of orthonormal basis, the Kähler form is $J^1 = e^1 \wedge e^2 - e^3 \wedge e^4$. We choose $H = 1$, so $f = \frac{a^2 g^4}{2}$. Now, however, we will take J^2 and J^3 to be u-dependent. Defining

$$L^2 = e^1 \wedge e^4 - e^2 \wedge e^3, \quad L^3 = e^1 \wedge e^3 + e^2 \wedge e^4, \tag{6.35}$$

we make the following choice for J^2 and J^3:

$$J^2 = \cos 2u L^2 + \sin 2u L^3, \tag{6.36}$$

$$J^3 = -\sin 2u L^2 + \cos 2u L^3. \tag{6.37}$$

Hence,

$$\psi = -J^1. \tag{6.38}$$

Our system of equations reduces to

$$\tilde{d}(K - J^1) = 0, \tag{6.39}$$

$$\tilde{d}(K + J^1) = 0, \tag{6.40}$$

$$(K - J^1) \wedge F = 0, \tag{6.41}$$

$$\tilde{\nabla}\mathcal{F} = 4(K^2 + (J^1)^2). \tag{6.42}$$

Equations (6.39)-(6.41) are solved by

$$K = e^1 \wedge e^2 + e^3 \wedge e^4, \tag{6.43}$$

and (6.42) is

$$\tilde{\nabla}^2\mathcal{F} = 16, \tag{6.44}$$

which is solved by

$$\mathcal{F} = 4\mathbf{x}^2. \tag{6.45}$$

The full solution is then

$$ds^2 = 2du(dv + 2\mathbf{x}^2 du) - d\mathbf{x}_2^2 - a^2 d\Omega_2^2, \tag{6.46}$$

$$G = \frac{4}{a^2 g^2} du \wedge dx^1 \wedge dx^2, \tag{6.47}$$

$$F = \frac{1}{ga^2} e^3 \wedge e^4, \tag{6.48}$$

$$e^{\sqrt{2}\phi} = \frac{a^2 g^2}{2}. \tag{6.49}$$

Since $K \neq 0$ this solution preserves one quarter supersymmetry, and to our knowledge it has not been given previously.

6.2 Non-twisting solutions of the $SU(2)$ theory

In analysing the $SU(2)$ theory it is convenient to find expressions for R_{ij} and R in terms of $\mathfrak{R}^a$ and $\mathfrak{R}$. Defining

$$\hat{\nabla}_i J^a_{jk} = \tilde{\nabla}_i J^a_{jk} + g\epsilon^{abc} A^b_i J^c_{jk}, \tag{6.50}$$

we have

$$[\hat{\nabla}_i, \hat{\nabla}_j] J_{mn} = 0, \tag{6.51}$$

which implies that

$$R_{ijmn} = \frac{1}{3}(J^{ak}_i J^{a\,l}_j R_{klmn} + 2J^a_{ij}\mathfrak{R}^a_{mn}), \tag{6.52}$$

on employing (6.3). From this we find

$$J^{ak}_i R_{kj} = \mathfrak{R}^a_{ij} - \epsilon^{abc} J^{bk}_i \mathfrak{R}^c_{kj}, \tag{6.53}$$

and hence that

$$\mathfrak{R} = \frac{1}{3}R, \tag{6.54}$$

$$\mathfrak{R}^a_{ij}\mathfrak{R}^{aij} = R_{ij}R^{ij} + \epsilon^{abc}\mathfrak{R}^{aj}_i \mathfrak{R}^b_{jk} J^{cik}. \tag{6.55}$$

On employing (6.3) again we find

$$\mathfrak{R}^a_{ij}\mathfrak{R}^{aij} = R_{ij}R^{ij} - \frac{1}{6}R^2, \tag{6.56}$$

and thus equation (6.6) is

$$\tilde{\nabla}^2 R = R^2 - 3R_{ij}R^{ij}, \tag{6.57}$$

for the $SU(2)$ theory. This equation is precisely the requirement that the Weyl anomaly for $\mathcal{N} = 4$ $U(N)$ SYM vanishes on the base. Whether this is a mere coincidence or has some deeper significance is unclear.

In fact, the Yang-Mills field strengths are precisely the curvatures of the left-hand spin bundle of the base (see eg [21]). The base must again have positive scalar curvature. Furthermore for supersymmetry we must only allow bases such that the anti-self dual parts of the curvatures of the left hand spin bundle are of the specific form

$$\mathfrak{R}^{a(-)} = \frac{R}{12}J^a, \tag{6.58}$$

from (6.3). In particular, Kähler bases are excluded, since then the anti selfdual parts of the curvatures are all proportional to the Kähler form, and this is inconsistent with super-symmetry. We note in passing that (6.58) is reminiscent of a quaternionic Kähler manifold in higher dimensions - the quaternionic Kähler condition is vacuous in four dimensions - though in the case at hand, the selfdual parts of the curvatures may (and indeed for a solution of (6.57) must) be nonzero.

6.2.1 Base $\mathbb{R} \times \mathcal{M}^3$

Equation (6.57) admits solutions of the form $\mathbb{R} \times \mathcal{M}^3$, with $\mathcal{M}^3$ Einstein. Choosing $\mathcal{M}^3$ to be an S^3 with round metric (more generally we could have the lens space $S^3/\mathbb{Z}_p$), and writing the base metric as

$$ds^2 = dx^2 + \frac{a^2}{4}\left((\sigma_R^1)^2 + (\sigma_R^2)^2 + (\sigma_R^3)^2\right), \tag{6.59}$$

we satisfy the constraint (6.58). We choose the J^i as in (6.29) with

$$e^1 = dx, \quad e^2 = \frac{a}{2}\sigma_R^1, \quad e^3 = \frac{a}{2}\sigma_R^3, \quad e^4 = \frac{a}{2}\sigma_R^2, \tag{6.60}$$

and we also choose $H = 1$, $f = \frac{a^2 g^4}{2}$, so that

$$e^{\sqrt{2}\phi} = \frac{a^2 g^2}{2}. \tag{6.61}$$

Then $G = \omega_F = 0$, and taking $\mathcal{F} = 0$, the solution is

$$ds^2 = dt^2 - d\mathbf{x}_2^2 - a^2 d\Omega_3^2,$$
$$F^a = -\frac{1}{8g}\epsilon^{abc}\sigma_R^b \wedge \sigma_R^c, \tag{6.62}$$

which is the Yang-Mills analogue of the Salam-Sezgin model; the S^3 is supported by a sphaleron (the Yang-Mills potentials are given by $A^a = -(2g)^{-1}\sigma_R^a$) and the solution preserves 1/4 supersymmetry, which is the maximum possible in the $SU(2)$ theory. We may easily find the Yang-Mills analogue of the compactifying pp-wave found in the U(1) theory; it is

$$ds^2 = 2du(dv + (3 + 4\tanh^2 u)x^2 du) - dx^2 - \cosh^2 u\, d\Omega_3^2,$$
$$F^a = -\frac{1}{8g}\epsilon^{abc}\sigma_b \wedge \sigma_c, \quad e^{\sqrt{2}\phi} = \frac{g^2}{2}\cosh^2 u, \quad G = 0. \tag{6.63}$$

6.2.2 Dyomeronic black string and $AdS_3 \times S^3$

Now we will show that the $SU(2)$ theory admits a black string solution with dyonic three form charges and a meron on the transverse space. It is straightforward to verify that the metric

$$ds^2 = dr^2 + \frac{a^2 r^2}{4}\sigma_R^a \sigma_R^a \tag{6.64}$$

is a (singular) positive scalar curvature solution of (6.57) and (6.58) when $a^2 < 1$. Again we choose the J^i as in (6.29) with

$$e^1 = dr, \quad e^2 = \frac{ar}{2}\sigma_R^1, \quad e^3 = \frac{ar}{2}\sigma_R^3, \quad e^4 = \frac{ar}{2}\sigma_R^2. \tag{6.65}$$

Then, the nonzero vielbein components of the Riemann tensor of the base are

$$R_{3434} = R_{2323} = R_{2424} = \frac{1}{a^2 r^2}(1 - a^2), \tag{6.66}$$

and the scalar curvature is

$$R = \frac{6}{a^2 r^2}(1 - a^2).$$ (6.67)

Let us take f to be given by

$$f = \frac{a^2 g^4}{2(1 - a^2)}\left(Q_1 + \frac{Q_2}{r^2}\right),$$ (6.68)

where (for a nonsingular six dimensional solution) we require $Q_1, Q_2 > 0$. Then

$$H = \frac{1}{r}\sqrt{Q_1 + \frac{Q_2}{r^2}},$$ (6.69)

$$e^{2\sqrt{2}\phi} = \frac{a^4 g^4}{4(1 - a^2)^2}(r^2 Q_1 + Q_2).$$ (6.70)

Let us take $K = \mathcal{F} = 0$. Then the F^a are given by

$$F^a = -\frac{(1 - a^2)}{8g}\epsilon^{abc}\sigma_R^b \wedge \sigma_R^c.$$ (6.71)

The metric is

$$ds^2 = \frac{r}{\sqrt{Q_1 + \frac{Q_2}{r^2}}}\eta_{\mu\nu}dx^\mu dx^\nu - \frac{1}{r}\sqrt{Q_1 + \frac{Q_2}{r^2}}(dr^2 + \frac{a^2 r^2}{4}\sigma_R^a \sigma_R^a),$$ (6.72)

and G is determined by H. The metric is everywhere nonsingular, and has a horizon at $r = 0$. In the near horizon limit $Q_1 \to 0$, defining $r = \sqrt{Q_2}u^{-1}$, the metric becomes

$$ds^2 = \sqrt{Q_2}\left(\frac{1}{u^2}(\eta_{\mu\nu}dx^\mu dx^\nu - du^2) - \frac{a^2}{4}\sigma_R^a \sigma_R^a\right),$$ (6.73)

which is $AdS_3 \times S^3$, the radius of curvature of the AdS factor being greater than that of the S^3. We may recover the $R^{1,2} \times S^3$ solution by setting the 3-form flux to zero in the limit $Q_2 \to \infty$, $a^2 \to 0$, $\sqrt{Q_2}a^2$ fixed.

6.3 u-independent solutions

In the interests of completeness we will write down the system of equations determining the general u-independent solution. Equations (6.1)-(6.4) remain valid, and the remaining equations reduce to

$$\tilde{d} \star_4 \tilde{d}(\Re e^{2\sqrt{2}\phi}) = \frac{2g^4}{\Re}(K - H\mathcal{G}) \wedge \tilde{d}\beta,$$ (6.74)

$$\tilde{d} \star_4 \tilde{d}\Re = 2\Re^a \wedge \Re^a$$
$$- \frac{2g^4}{e^{2\sqrt{2}\phi}\Re}(K + H\mathcal{G}) \wedge \tilde{d}\beta,$$ (6.75)

$$\tilde{d}\left(\frac{1}{\Re}[K - H\mathcal{G}]\right) = 0,$$ (6.76)

$$\tilde{d}\left(\frac{e^{-2\sqrt{2}\phi}}{\Re}[K + H\mathcal{G}]\right) = 0,$$ (6.77)

$$\Re^a \wedge (K - H\mathcal{G} + (\tilde{d}\omega)^-) = 0,$$ (6.78)

$$-\frac{1}{2}\star_4 \tilde{d}\star_4 \tilde{d}\mathcal{F} = 2H^{-2}(K^2 - H^2\mathcal{G}^2).$$ (6.79)

We note that these may be solved as follows: take $\omega = K = \mathcal{F} = 0$, the base to be given by a solution of (6.16) or (6.57) as appropriate, and β to be a form with self-dual field strength on the base but otherwise arbitrary. We have been unable to find any other nonsingular solutions.

7. Solutions with enhanced supersymmetry

In this section we will obtain further constraints on the bosonic fields implied by enhanced supersymmetry - that is, we consider solutions preserving one half or one quarter super-symmetry in the $U(1)$ and $SU(2)$ theories respectively. Given two linearly independent Killing spinors ϵ, ϵ', with associated Killing vectors V, V', we always have

$$V_\mu \Gamma^\mu \epsilon = V'_\mu \Gamma^\mu \epsilon' = 0. \tag{7.1}$$

However there is no reason why we should have $V_\mu \Gamma^\mu \epsilon' = 0$ or $V'_\mu \Gamma^\mu \epsilon = 0$ (see 5.1 of [12] for further discussion of this point). Thus we will relax the condition $\Gamma^+ \epsilon = 0$ in this section, though we may of course still employ our general ansatz.

7.1 U(1) solutions preserving one half supersymmetry

Let us define

$$F_B = \tilde{F} - \frac{g}{4} H e^{-\sqrt{2}\phi} J^1. \tag{7.2}$$

Now we know from our general ansatz that

$$F = e^+ \wedge \omega_F + F_B. \tag{7.3}$$

Then, the vanishing of $\delta\lambda$ implies that

$$(\omega_{Fi}\Gamma^{+i} + F_{Bij}\Gamma^{ij} + 2g e^{-2\sqrt{2}\phi} T^1)\epsilon = 0. \tag{7.4}$$

Employing $(\Gamma^+)^2 = 0$, $\Gamma^+ \epsilon \neq 0$ then gives

$$F_{Bij}\Gamma^{ij}\epsilon = -2g e^{-2\sqrt{2}\phi} T^1 \epsilon, \tag{7.5}$$

$$\omega_{Fi}\Gamma^i \epsilon = 0. \tag{7.6}$$

We impose the projection (7.5), and require that this is the only algebraic constraint on the Killing spinor. Hence we must impose $\omega_F = 0$. Next we note that

$$(F_{Bij}\Gamma^{ij} + 2g e^{-\sqrt{2}\phi} T^1)\delta\lambda = 0, \tag{7.7}$$

so

$$(-F_{Bij}F_{Bkl}\Gamma^{ijkl} + 2F_{Bij}F_B^{ij} - g^2 e^{-2\sqrt{2}\phi})\epsilon = 0, \tag{7.8}$$

and hence

$$F_B \wedge F_B = 0, \tag{7.9}$$

$$F_{Bij}F_B^{ij} = \frac{g^2}{2} e^{-\sqrt{2}\phi}, \tag{7.10}$$

and so F_B is a decomposable two form.

For the vanishing of $\delta\chi$ to imply no further restrictions on the Killing spinor, we clearly must have $\phi = $ constant, $G^- = 0$. Then given (7.9), the three form field equations are implied by the Bianchi identities, and (7.10) is equivalent to the dilaton field equation. Requiring $[D_\mu, \Delta_F]\epsilon = 0$ implies

$$(\nabla_\mu F_{\nu\sigma} + 2e^{\sqrt{2}\phi}G^+_{\mu\nu}{}^\tau F_{\sigma\tau})\Gamma^{\nu\sigma}\epsilon = 0, \tag{7.11}$$

and hence that

$$\nabla_\mu F_{\nu\sigma} = -2e^{\sqrt{2}\phi}G^+_{\mu[\nu}{}^\tau F_{\sigma]\tau}, \tag{7.12}$$

which (for constant dilaton) implies the field equations for F. Given the two form Bianchi identities we also find

$$G^+_{[\mu\nu}{}^\tau F_{\sigma]\tau} = 0. \tag{7.13}$$

Next $[D_\mu, D_\nu]\epsilon = 0$ together with $\delta\lambda = 0$ implies

$$(R_{\mu\nu\sigma\tau} - e^{\sqrt{2}\phi}\nabla_\mu G^+_{\nu\sigma\tau} + e^{\sqrt{2}\phi}\nabla_\nu G^+_{\mu\sigma\tau}$$
$$- 2e^{2\sqrt{2}\phi}G^+_{\mu\sigma}{}^\rho G^+_{\nu\tau\rho} + 2e^{\sqrt{2}\phi}F_{\mu\nu}F_{\sigma\tau})\Gamma^{\sigma\tau}\epsilon = 0, \tag{7.14}$$

and hence the expression in parentheses must vanish. Antisymmetrising this expression on ν, σ and τ and employing (7.9) together with the Bianchi identity and self-duality of G^+, we find

$$\nabla G^+ = 0. \tag{7.15}$$

Thus G^+ is parallel with respect to the Levi-Civita connection. We therefore find for the Riemann tensor that

$$R_{\mu\nu\sigma\tau} = -2(e^{2\sqrt{2}\phi}G^+_{\mu[\sigma}{}^\alpha G^+_{\tau]\nu\alpha} + e^{\sqrt{2}\phi}F_{\mu\nu}F_{\sigma\tau}). \tag{7.16}$$

Given equation (7.10) and that $\phi = $ constant, $G^- = 0$, this implies that all the Einstein equations are satisfied.

We will not attempt a complete analysis of solutions with one half supersymmetry, imposing these additional constraints on our general ansatz. Rather we shall show the uniqueness of the Salam-Sezgin vacuum among one half supersymmetric solutions with vanishing three form flux, and give a brief discussion of a special class of solutions.

7.1.1 $G^+ = 0$

When $G^+ = 0$, the 6 dimensional Riemann tensor is parallel with respect to the Levi-Cicita connection and the geometry is locally symmetric. Since F is decomposable and its only nonvanishing components are spacelike, (7.16) implies that the solution is locally isometric to $\mathbb{R}^{1,3} \times \mathcal{M}_2$, where $\mathcal{M}_2$ is a symmetric Riemannian two manifold. Equation (7.10) then implies that $\mathcal{M}_2 = S^2$. Thus any solution with vanishing three form flux which preserves one half supersymmetry is locally isometric to $\mathbb{R}^{1,3} \times S^2$. If we assume in addition simple connectedness then any such solution is in fact isometric to $\mathbb{R}^{1,3} \times S^2$.

7.1.2 One half supersymmetric solutions with Kähler base

When the base is Kähler, we know that F equals the Ricci form of the base. The decomposability of the Ricci form implies that the Ricci tensor of the base has two zero eigenvalues. Then equation (6.16) implies that the other pair of eigenvalues (which are necessarily equal since the base is Kähler) is harmonic; in order that the base have positive scalar curvature they must also be everywhere positive. If the nonzero pair of eigenvalues are in fact constant, and we assume that the base is compact, then according to [22] the base is locally the product $T^2 \times S^2$. When the nonzero pair of eigenvalues is not constant, we can offer no such general result. Instead we will consider a specific example; consider a base equipped with the metric

$$ds^2 = \frac{dr^2}{W^2(r)} + \frac{r^2}{4}((\sigma_R^1)^2 + (\sigma_R^2)^2 + W^2(r)(\sigma_R^3)^2). \tag{7.17}$$

Choosing the vierbeins as

$$e^1 = \frac{r}{2}\sigma_R^1, \quad e^2 = \frac{r}{2}\sigma_R^2, \quad e^3 = \frac{rW}{2}\sigma_R^3, \quad e^4 = W^{-1}dr, \tag{7.18}$$

we find the following for the vierbein components of the Ricci tensor:

$$R_{11} = R_{22} = -\frac{2WW'}{r} - \frac{4W^2}{r^2} + \frac{4}{r^2}, \tag{7.19}$$

$$R_{33} = R_{44} = -WW'' - (W')^2 - \frac{5WW'}{r}. \tag{7.20}$$

Now requiring $R_{11} = 0$ gives $W^2 = 1 - ar^{-4}$, namely Eguchi-Hanson, which is Ricci flat and thus not an allowed base. Therefore, we impose $R_{33} = 0$. On employing $R_{33} = 0$, we find that $R' = -2r^{-1}R$, hence $R = ar^{-2}$. Next, imposing $\nabla^2 R = 0$, or

$$R' = \frac{b}{r^3 W^2}, \tag{7.21}$$

we get $W = $ constant, which solves $R_{33} = 0$. Finally for a positive scalar curvature Riemannian metric, we must have $0 < W < 1$. This is therefore the unique base of the form (7.17) which induces a one half supersymmetric solution. This base was employed in the construction of the $U(1)$ dyonic string, which is one quarter supersymmetric, as it does not have constant dilaton. To get a one half supersymmetric solution, we take $f = cR$ to obtain the AdS_3 times a squashed S^3 solution of [7], which arises as the near horizon limit of the dyonic string.

7.2 $SU(2)$ solutions preserving one quarter supersymmetry

To obtain solutions of the $SU(2)$ theory preserving one quarter supersymmetry, we must again relax the condition that $\Gamma^+\epsilon = 0$. As before, this implies that $\omega_F^a = 0$. We must then impose any two of the projections $\delta\lambda^a = \delta\lambda^b = 0$, together with $[\Delta_{F^a}, \Delta_{F^b}]\epsilon \sim \epsilon^{abc}\Delta_{F^c}\epsilon$. Thus we find

$$F_{B[i}^{a}{}^{k}F_{Bj]k}^{b} = \frac{g}{4}e^{-\sqrt{2}\phi}\epsilon^{abc}F_{Bij}^{c}. \tag{7.22}$$

Now we also have

$$(F^a_{Bij}\Gamma^{ij} + 2ge^{-\sqrt{2}\phi}T^a)\delta\lambda^b = 0, \tag{7.23}$$

or

$$\left(\frac{1}{2}F^{aij}_B F^b_{Bkl}([\Gamma_{ij},\Gamma^{kl}] - \{\Gamma_{ij},\Gamma^{kl}\}) + g^2 e^{-2\sqrt{2}\phi}(2\epsilon^{abc}T^c - \delta^{ab})\right)\epsilon = 0. \tag{7.24}$$

Using (7.22), this becomes

$$(-F^a_{Bij}F^b_{Bkl}\Gamma^{ijkl} + 2F^a_{Bij}F^{bij}_B - g^2 e^{-2\sqrt{2}\phi}\delta^{ab})\epsilon = 0, \tag{7.25}$$

and thus

$$F^a_B \wedge F^b_B = 0, \tag{7.26}$$

$$F^a_{Bij}F^{bij}_B = \frac{g^2}{2}e^{-2\sqrt{2}\phi}\delta^{ab}. \tag{7.27}$$

By arguments identical to those employed in the analysis of the $U(1)$ theory, we find that the following constraints, together with (7.22), (7.26) and (7.27), are implied by the requirement of one quarter supersymmetry:

$$\phi = const, \tag{7.28}$$

$$G^- = 0, \tag{7.29}$$

$$R_{\mu\nu\sigma\tau} = -2(e^{2\sqrt{2}\phi}G^+_{\mu[\sigma}{}^\alpha G^+_{\tau]\nu\alpha} + e^{\sqrt{2}\phi}F^a_{\mu\nu}F^a_{\sigma\tau}), \tag{7.30}$$

$$\nabla G^+ = 0, \tag{7.31}$$

$$\nabla_\mu F^a_{\nu\sigma} + 2g\epsilon^{abc}A^b_\mu F^c_{\nu\sigma} = -2e^{\sqrt{2}\phi}G^+_{\mu[\nu}{}^\tau F^a_{\sigma]\tau}, \tag{7.32}$$

and given the Bianchi identities of the forms, all field equations are satisfied. We could impose these additional constraints on our general ansatz to more fully characterise solutions preserving one quarter supersymmetry. However we will merely observe that they are satisfied by our $AdS_3 \times S^3$ and $\mathbb{R}^{1,2} \times S^3$ solutions. The latter is the unique solution with vanishing three form flux which preserves one quarter supersymmetry. To see this, we note that when $G = 0$, (7.30) and (7.32) imply that $R_{\mu\nu\sigma\tau}$ is parallel with respect to ∇, and hence the geometry is locally symmetric. Further since the F^a only have nonzero components on the base, the solution is necessarily locally isometric to $\mathbb{R}^{1,1} \times \mathcal{M}_4$, where $\mathcal{M}_4$ is a symmetric positive scalar curvature non Kähler Reimannian manifold. By direct inspection of equation (6.57), we see that the only possibility for the base is $\mathbb{R} \times S^3$. Thus a solution with vanishing three form flux which preserves one quarter supersymmetry is necessarily locally isometric to $\mathbb{R}^{1,2} \times S^3$, with isometry if we assume simple connectedness.

8. Penrose Limits

In this section we will define the Penrose limits of the $U(1)$ and $SU(2)$ theories and employ our definition to derive nonabelian pp-wave solutions, closely following [23], [24]. In the neighbourhood of a segment of a null geodesic containing no conjugate points, we may introduce local null coordinates U, V, X^i such that the metric takes the form

$$ds^2 = dV\left(dU + \alpha dV + \rho_i dX^i\right) - C_{ij}dX^i dX^j. \tag{8.1}$$

We may also choose a gauge such that the one and two form potentials satisfy

$$A_U^a = B_{U\alpha} = 0. \tag{8.2}$$

As usual, we introduce a positive real constant Ω and rescale the coordinates:

$$U = u, \quad V = \Omega^2 v, \quad X^i = \Omega x^i. \tag{8.3}$$

We thus obtain an Ω-dependent family of fields $g_{\mu\nu}(\Omega)$, $\phi(\Omega)$, $A_\mu^a(\Omega)$, $B_{\mu\nu}(\Omega)$. We define the familiar rescaled fields, distinguished by an overbar:

$$\bar{g}_{\mu\nu}(\Omega) = \Omega^{-2} g_{\mu\nu}(\Omega), \tag{8.4}$$

$$\bar{\phi}(\Omega) = \phi(\Omega), \tag{8.5}$$

$$\bar{A}_\mu^a(\Omega) = \Omega^{-1} A_\mu^a(\Omega), \tag{8.6}$$

$$\bar{B}_{\mu\nu}(\Omega) = \Omega^{-2} B_{\mu\nu}(\Omega). \tag{8.7}$$

However under these rescalings neither the Lagrangian nor the supersymmetry variations of the theory transform homogeneously; the terms which do not transform appropriately are the nonlinear $gA \wedge A$ terms in the Yang-Mills field strengths, the $gA \wedge A \wedge A$ terms in the three form field strength and the $ng^2 e^{-\sqrt{2}\phi}$ term in the Lagrangian (which is also present in the $U(1)$ theory). Thus if we want a well defined Penrose limit which takes supersymmetric solutions into supersymmetric solutions we must rescale the gauge coupling constant according to

$$\bar{g} = \Omega g, \tag{8.8}$$

so that the Yang-Mills and three form field strengths are rescaled according to

$$\bar{F}^a(\Omega) = \Omega^{-1} F^a(\Omega), \tag{8.9}$$

$$\bar{G}(\Omega) = \Omega^{-2} G(\Omega), \tag{8.10}$$

and then the Lagrangian and supersymmetry transformations transform homogeneously. The Penrose limit consists of taking the limit of the barred fields (and coupling) as $\Omega \to 0$; in particular, we have $\bar{g} \to 0$, so we will obtain pp-wave solutions of the ungauged theory. The solutions of the SU(2) theory are non-abelian pp-waves of the same kind as those described in 4-dimensions in [25].

8.1 An example: the Penrose limit of $AdS_3 \times S^3$

Defining R_1, R_2 to be the radii of curvature of the AdS_3 and S^3 factors respectively, and $a = R_2/R_1$, let us write our $AdS_3 \times S^3$ solution in the following form:

$$R_1^{-2} ds^2 = dt^2 - \sin^2 t \left[\frac{dr^2}{1+r^2} + r^2 d\theta^2 \right]$$
$$- a^2 (d\chi^2 + \sin^2 \chi (d\lambda^2 + \sin^2 \lambda d\psi^2)), \tag{8.11}$$

$$G = \frac{2(1-a^2)}{R_1 R_2^2 g^2} (Vol_{R_1}(AdS_3) + Vol_{R_2}(S^3)), \tag{8.12}$$

$$F^1 = \frac{1-a^2}{g} \sin^2 \chi \sin \lambda d\lambda \wedge d\psi, \; F^2 = \frac{1-a^2}{g} \sin \chi d\chi \wedge d\lambda, \tag{8.13}$$

$$F^3 = \frac{1-a^2}{g} \sin \chi \sin \lambda d\psi \wedge d\chi, \; e^{\sqrt{2}\phi} = \frac{R_2^2 g^2}{2(1-a^2)}. \tag{8.14}$$

Here $Vol_{R_i}(\mathcal{M})$ means the volume form of the manifold $\mathcal{M}$ with radius of curvature R_i. Let us now introduce coordinates u, v according to

$$u = t + a\chi, \quad v = t - a\chi, \tag{8.15}$$

and take the Penrose limit along the null geodesic parametrised by u. We obtain the metric

$$R_1^{-2}d\bar{s}^2 = dudv - \sin^2(u/2)g(\mathbb{R}^2) - a^2\sin^2(u/2a)g(\mathbb{R}^2). \tag{8.16}$$

Let us introduce coordinates y^i, $i = 1,..,4$ such that the metric takes the form

$$R_1^{-2}d\bar{s}^2 = dudv - \sum_1^4 \frac{\sin^2(\lambda_i u)}{(2\lambda_i)^2}dy^i dy^i, \tag{8.17}$$

where

$$\lambda_i = \begin{cases} \frac{1}{2}, & i = 1,2 \\ \frac{1}{2a}, & i = 3,4. \end{cases} \tag{8.18}$$

Finally let us change coordinates to

$$x^- = R_1\Big(v + \frac{1}{4}\sum_i y^i y^i \frac{\sin(2\lambda_i u)}{2\lambda_i}\Big), \quad x^+ = \frac{R_1 u}{2}, \quad x^i = R_1 y^i \frac{\sin(\lambda_i u)}{2\lambda_i}, \tag{8.19}$$

so that the metric becomes

$$d\bar{s}^2 = 2dx^+\Big(dx^- + 2\Big(\sum_i \Big(\frac{\lambda_i}{R_1}\Big)^2 x^i x^i\Big)dx^+\Big) - dx^i dx^i. \tag{8.20}$$

For the fluxes we obtain

$$\bar{G} = \frac{2(1-a^2)}{R_1 R_2^2 g^2}dx^+ \wedge (dx^1 \wedge dx^2 + dx^3 \wedge dx^4), \tag{8.21}$$

$$\bar{F}^1 = 0, \tag{8.22}$$

$$\bar{F}^2 = \frac{1-a^2}{R_2^2 g}\frac{dx^+}{\sqrt{(x^3)^2 + (x^4)^2}} \wedge (x^3 dx^3 + x^4 dx^4), \tag{8.23}$$

$$\bar{F}^3 = \frac{1-a^2}{R_2^2 g}\frac{dx^+}{\sqrt{(x^3)^2 + (x^4)^2}} \wedge (x^4 dx^3 - x^3 dx^4). \tag{8.24}$$

In the limit $R_1 \to \infty$, R_2 fixed, $G \to 0$ and the metric becomes that of $CW_4 \times \mathbb{R}^2$. We could have obtained this directly by taking the Penrose limit of the $\mathbb{R}^{1,2} \times S^3$ solution. Finaly we note that the vanishing of $\bar{F}^1$ is compatible with supersymmetry; after taking the Penrose limit, the supersymmetry variation of the gauginos is

$$\delta\bar{\lambda}^a = -\frac{1}{2\sqrt{2}}e^{-\bar{\phi}/\sqrt{2}}\bar{F}^a_{\mu\nu}\Gamma^{\mu\nu}\epsilon. \tag{8.25}$$

9. Conclusions

We have found the most general supersymmetric ansatz for the six dimensional chiral gauged $U(1)$ and $SU(2)$ theories, and explored the geometrical structure of the solutions. Our results display both the strengths and weaknesses of this general approach to finding supersymmetric solutions. Because of the difficulties in solving equations (6.16) and (6.57) in particular, we cannot claim to have achieved the same degree of completeness in classifying all supersymmetric solutions as was attained in [11] or [12], for example. However despite the fact that the theories we have examined are considerably more complicated, they are still to some degree tractable. Most encouragingly, we have demonstrated that the nonabelian theory is at least no more intractable than the abelian one, and one might hope that a similar approach applied to other nonabelian gauged supergravities would be fruitful, and might allow the construction of interesting new string/ M-theory solutions. We have given a (rather implicit) classification of solutions of both theories with enhanced supersymmetries. We have also explored the Penrose limits of these gauged supergravities, and shown that they yield pp-wave solutions of the ungauged theories.

Perhaps the greatest advantage in employing the G-structures approach to the solution of supergravities is in the geometric insight one obtains into the form of the supersymmetric solutions. For example in the non-twisting case, finding solutions of our non-abelian gauged supergravity in six dimensions essentially reduces to a problem in pure four dimensional Riemanian geometry, namely the solution of (6.57) subject to the constraint (6.58). Conceptually this is an enormous simplification. However from a practical point of view it is still a difficult problem, and we have only succeeded in finding two explicit solutions.

Since the appearance of [10], we know how to embed the Salam-Sezgin model in string theory. It would be of interest to find the string theory realisation of the black string solutions of the theories we have studied, and to study the holography of the AdS_3 solutions.

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